

Peter Gärdenfors

GROUP DECISION THEORY

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This essay is a summary of four articles which I have written in compliance with the requirements for the Swedish examination "filosofie doktor". The papers and the abbreviations which appear in the text are:

- /AP/ Assignment problem based on ordinal preferences. Management Science, vol. 20, no. 3 (1973), pp. 331 - 340.
- /PVF/ Positionalist voting functions. Theory and Decision, vol. 4 (1973), pp. 1 - 24.
- /MM/ Match-making: Assignments based on bilateral preferences. Working Paper No. 13, The Mattias Fremling Society, Lund 1974 (mimeographed).
- /QP/ Qualitative probability as an intensional logic. Princeton 1973 (mimeographed).

In writing these articles I have benefited from discussions with many philosophers. Above all, I mention my advisor and teacher Bengt Hansson who has been my intellectual guide from the very first paper I have written. My interest in group decision theory was stimulated after reading his papers on the topic. Our discussions during lunch-hours not only resulted in a next to infinite number of scribbled napkins, but provided as well the necessary testing of my ideas. His influence on my work has been so great that it would not be improper to list him as a co-author for all of the papers, were it not that he would then become liable for the mistakes therein.

I have also profited from the advice and support of Sören Halldén.

Thanks to a fellowship from the American-Scandinavian Foundation, which I gratefully acknowledge, I was able to visit Princeton University during the academic year 1973 - 1974. The excellent working conditions and the stimulating discussions with the teachers and the graduate students there made it possible for me to write /QP/ and /MM/ and to outline this summary during my stay at Princeton. In particular I would like to thank David Lewis of Princeton and Zoltan Domotor and Richard Jeffrey of the University of Pennsylvania for their encouragement and valuable help.

I would also like to acknowledge Stanley Noval for the efforts he has made in correcting my brutalized English and Barbro Löwegren for typing much of the manuscript. Without their help this thesis would hardly have been legible.

Forsakar
Midsummer Day 1974
P. G.

... I have done enough to demonstrate the correctness of my details. The defects, ... , are merely basic and fundamental.

Ambrose Bierce

I. INTRODUCTION

Since midcentury, the theory of decision has undergone a swift and broad development and has come to occupy a prominent position in the social and behavioural sciences. Decision theory has had such rich applications in economics, psychology and political science that one tends to overlook its basic grounding in mathematics and philosophy. Perhaps because the principal contributors have been so widely separated in discipline and training, there is no single 'theory of decision' which can be set out in a coherent framework. Instead, there are several theories, each tailored to suit the needs of the discipline to which it is handmaiden.

Many of the concepts used in decision theory were first introduced by economists, long concerned with the choices of consumers and of producers in a market economy. The use of formal decision theory has made it possible to prove a variety of theorems about the existence and stability of economic systems and to investigate the behaviour of economic agents under conditions of uncertainty and risk. Decision theory is now regarded by economists as a fundamental analytical tool in that discipline.

Psychologists too have concerned themselves with decision theory and its possibilities for explaining and predicting certain kinds of human behaviour. In contrast with economists, they are less interested in normative questions such as how one ought to make decisions, but rather in such matters as how one might describe the decision process.

Also within the social sciences, decision theory has sparked great interest among political theorists concerned with parliamentary voting procedures and other problems of group decision. Political scientists have long sought to specify voting methods capable of accounting for individual values in as 'fair' a way as possible. Arrow's voting paradox came as a shock to the discipline and the development of the theory of voting in the ensuing years can be described as "explosive".

However fruitful the use of decision theory in economics, psychology and political science, the most intense progress in the field has occurred within the realm of mathematical statistics.

Recognition of the purely mathematical character of decision theory dates from 1944, with the publication of von Neumann and Morgenstern's /62/. As a special branch of decision theory, the theory of games has penetrated to some degree the social sciences, but has had its greatest impact among mathematicians and statisticians.

With all the activity in these several disciplines, the essential philosophical aspects of decision theory have sometimes been overlooked. In most decision theory, there is an omnipresent concept of utility. With its earliest roots in Bentham's "felicity calculus", decision theory has always been deeply imbued with utilitarian principles. In a broad sense, philosophical interest in group decision theory can be said to center on the operational meaning of the basic utilitarian rule of "the greatest good for the greatest number". Beyond this, group decision theory has in my opinion a great deal to contribute to the philosophical analysis of such concepts as 'democracy', 'impartiality', 'fairness' and perhaps even 'justice'.

Figuratively speaking, decision theory supplies the practical logician with a stock of materials suited to his precise analytical tools. From a technical point of view the logics of preferences and subjective probabilities are branches of formal logic, which can be studied per se or as a formal basis of a strict theory of expected utility.

As its title suggests, my principal concern in this essay is with group decision problems although my attention is not strictly limited to these matters. Chapter 2 briefly sets out a classification of some of the main branches of decision theory and shows how they may be fitted into a general pattern. In chapter 3, which is devoted to group decision processes, I spend some time arguing that many problems, formerly treated as 'individual' decision making, can be more coherently viewed as 'group' decision making. In the main part of the chapter, however, the problem of voting and some assignment problems are described and discussed. Apart from matters directly concerning group decision theory, chapter 4 briefly presents some results and problems concerning the logic of subjective probability and expected utility.

II. A CLASSIFICATION OF DECISION THEORY

1.

A decision, in its most general form, is a choice of one of the alternatives available in a given situation. A decision procedure is a rule which, for any situation, determines the alternative to be chosen. Decision procedures can be more technically described as choice functions which have situations as arguments and alternatives as values. For many situations there may not seem to be any reason to distinguish between two or more alternatives. In order to allow a degree of generality in the analysis, I shall therefore permit a choice function to select one or more of the alternatives. Any selection of alternatives will be called a choice set. From such a set the final alternative may then be chosen by some tie-breaking procedure, e.g. by flipping a coin.

I will denote the set of alternatives by A and the particular alternatives by x, y, z etc. It may in practice be a difficult problem to specify the elements of A , but this is a matter about which I will have little to say. However, by including alternatives as 'the decision is delayed to a later time' or 'no decision is reached' one can assure that some alternative is always chosen.

The set of situations will be denoted S and its members a, b, c etc. The situations are supposed to contain the information about the alternatives on which the decision is based. I will have more to say about the kind of information given by the situation in a subsequent section.

The following definition may now be introduced.

Definition 1.1: A choice function is a function from S to the set of non-empty subsets of A .

This definition is too broad to specify the class of 'rational' choice functions. In order to delimit the class of admissible functions one must introduce some axioms in addition to the definition above. Although there is no general agreement on which axioms are the correct ones for choice functions, there are certain minimal requirements¹. These are to a large extent dependent

on the nature of the decision problem. I will return to the relevant axioms in connection with specific problems.

The goal of decision theory is to characterize the class of 'rational' decision procedures, i.e. the class of choice functions which in any situation select the 'best' alternatives. One must, however, distinguish between normative and descriptive decision theory. A normative theory states the decision procedure to be used, if the decision maker is assumed to be 'rational'. A descriptive theory, on the other hand, serves as a basis for prediction and explanation of human behaviour in choice situations. There is of course no sharp distinction between normative and descriptive decision theories. A proposed normative theory is tested against our intuitive understanding of rational decisions; and most descriptive theories are heavily influenced by existing normative theories. In this work I will be concerned only with the normative aspects of decision theory, although I will occasionally use experimental evidence to illustrate the merits or drawbacks of certain conditions on choice functions.

The decision theory I will consider will be static in the sense that the choice functions do not account for temporal factors. I will therefore not concern myself with learning processes or in any other way account for the effect on decision processes of repeated experiments.

Furthermore I shall assume the choice functions to be deterministic in contrast to probabilistic approaches where a certain alternative (or a set of alternatives is chosen only with some degree of probability in a given situation. Note, however, that this is not the same as assuming that the outcome of the choice is known with certainty. The alternative to be chosen may be an act, the outcome of which depends on the true (and partially unknown) state of the world².

2.

I turn now to a closer description of what an alternative is and what kind of information about the alternatives is used in decision procedures. This will lead to the traditional distinction between decisions under certainty, under risk and under uncertainty.

In many decision situations the alternatives can be conceived of as acts, from among which the decision maker has to perform one. In such cases a rational decision maker has to deliberate upon the possible consequences or outcomes of the available acts to assure himself that he chooses the 'best' act. Though the decision maker presumably has some control over the variables which determine the outcome, he does not in general have complete control. Sometimes a chance event controls the outcome of a decision, sometimes part of the control is in the hands of other persons whose decisions, together with the decision maker's, establishes the outcome and sometimes both types of factors have influence. The theory which deals with decisions of several individuals with conflicting interests is called game theory. I will avoid here pure game theory by eliminating any direct reference to other persons' decisions. Instead I will introduce the uncertainty as to the consequences of an act by employing the notion of states of the world. Savage /54/ introduces this concept in the following manner³:

Term:	Definition:
the <u>world</u>	the object about which the person is concerned
a state (of the world)	a description of the world, leaving no relevant aspect undescribed
the <u>true</u> state (of the world)	the state that does in fact obtain, i.e. the true description of the world

The terminology is "brief, suggestive and in reasonable harmony with the usage of statistics and ordinary discourse" although not a definition in the strict sense⁴.

The concept of a 'state of the world' is obviously closely related to the notion of 'a possible world' as the term has been used in recent discussions of semantics for intensional logics⁵. The main difference seems to be that a possible world determines the truth or falsity of any proposition, while Savage's states only determine the truth value of the 'relevant' propositions. I will, however, not make any excursion into the metaphysics and ontology of possible worlds, but will be content to use the term more or less as a primitive concept.

I can now define an alternative as something which determines an outcome, depending on the state of the world.

Definition 2.1: Let $\underline{W}$ denote the set of all possible states of the world and $\underline{O}$ the set of possible outcomes. An alternative is a function from $\underline{W}$ to $\underline{O}$. The outcomes of an alternative x for a state $\underline{s}$ is denoted $x(s)$.

This is essentially the definition of an 'act' as presented by Savage. There is, however, the interesting possibility of including the statement that a certain outcome has occurred in the description of the state of the world. In such a case an alternative can simply be identified with a set of possible states of the world, where the outcome is a part of the information about each of the worlds. This is the approach taken by Jeffrey in /34/.

A crucial factor for a decision maker (and for a normative decision theory) is his knowledge about which state of the world is the true state. To begin with, one may distinguish two extremes:

- (i) The decision maker has no information (relevant to the decision problem) about the true state of the world.
- (ii) The decision maker has full information, in the sense that he knows (assumes) a probability distribution $\underline{P}$ over $\underline{W}$, the set of possible states, such that for any subset X of $\underline{W}$, $\underline{P}(X)$ represents the probability that the true state is in X .

Usually it is assumed that the alternative chosen does not have any influence on which state is the true state. If, however, one wants to account a situation in which the probability of a subset X of $\underline{W}$ is influenced by the fact that the alternative x was chosen, one should use the conditional probability $\underline{P}(X/x)$ instead of $\underline{P}(X)$ as a basis for a deliberation of the alternatives. Jeffrey's approach, identifying the alternatives with subsets of $\underline{W}$, makes such a complication unnecessary.

The probability measure $\underline{P}(X)$ is taken to represent the decision maker's subjective or personal degree of belief in the proposition that the true state of the world is in X .

It may happen that the probability distribution $\underline{P}$ gives the decision maker enough information to determine that each

alternative, independently of the state of the world, leads to a specific outcome. More formally, this occurs when for each alternative x_i there is an outcome o_i such that the probability of the set of states $\underline{s}$ for which $x_i(\underline{s}) = o_i$ equals one⁶. If this much information is assumed, it is commonly said that the decision is made under certainty. In all other cases of (ii), i.e. when there are some alternatives for which several outcomes are possible, we say that the decision is made under risk. Finally, if (i) obtains, the decision is traditionally said to be made under uncertainty.

Note that if the decision is made under certainty, then to each alternative there corresponds a unique outcome; and if two alternatives have the same outcome, it is of no importance for the decision problem whether they are regarded as distinct alternatives. Consequently, in decision under certainty the alternatives may be identified with the outcomes.

Between the two extremes (i) and (ii) there is a vast area of different degrees of partial information. It would be a fiction not to admit that most practical decision problems fall within this area. In most cases the decision maker has some information, which is relevant for the decision problem, about the probabilities of the different states, but it is almost only in games with coins and dice that he can reasonably be said to have full information. The area between the two extremes is also the province of statistical decision making. Lacking both space and knowledge, I am, however, unable to pursue these delicate matters.

3.

Heretofore, the alternatives have been regarded as an unstructured set. I now turn to a description of different ways of imposing a mathematical structure on the set of alternatives. I do not intend to provide an exhaustive analysis, but merely to indicate the major possibilities.

In much of the economic literature alternatives are interpreted as bundles of commodities which may be purchased by a consumer. If there are n different commodities, such a bundle

can be described by an n -dimensional vector $\underline{x} = (x_1, \dots, x_n)$, where the component x_i specifies the amount of commodity i to be consumed. Under this interpretation, the set of alternatives forms a (convex) subset of a vector space, which is bounded by the constraint that the total cost of a bundle must be less than a certain fixed amount. With this structure on the set of alternatives one can find intuitively reasonable axioms on a preference ordering (or a choice function)⁷ which cannot be formulated otherwise. A typical example is the axiom of nonsatiety which demands that if $x_i \geq y_i$ for every dimension i in the vectors $\underline{x}$ and $\underline{y}$ and $x_j > y_j$ for some j , then $\underline{x}$ is better than $\underline{y}$ (or, in terms of a choice function, $\underline{y}$ is not chosen).

Since the appearance of von Neumann and Morgenstern's /62/ considerable importance has been attached to sets of alternatives that form so-called mixture spaces. If one has two alternatives x and y , then a new alternative may be formed which consists of alternative x with probability α and y with probability $1 - \alpha$. This new alternative is called the α -mixture of x and y , and can be denoted by $x \alpha y$. The following axioms are introduced for this mixture operation:

- (1) if x and y are in $\underline{A}$, then $x \alpha y$ is in $\underline{A}$, for any α between 0 and 1
- (2) $x \alpha y = y (1-\alpha) x$
- (3) $(x \alpha y) \beta y = x \alpha \beta y$
- (4) $x 1 y = x$

As a typical condition for a preference ordering of the set of alternatives which form a mixture space I will just mention the following: if x is better than y , then $x \alpha y$ is better than y if $\alpha \neq 0$ and x is better than $x \alpha y$ if $\alpha \neq 1$.⁸

In Jeffrey /34/ the alternatives (the 'acts' in his terminology) are interpreted as propositions (which then can be handled as a set of possible worlds). Explaining this view, he writes:⁹

"where acts are characterized with sufficient accuracy by declarative sentences, we can conveniently identify the acts with the propositions that the sentences express. An act is then a proposition which it is within the agent's power to make true if he pleases,.....".

Once we have a set of propositions we can perform the usual logical operations conjunction, disjunction, negation etc. on them, ruled by a standard set of axioms. There are some problems with the interpretations of Jeffrey's view. For example the disjunction of two act-describing propositions may not be an act-describing proposition.

Whether we regard the alternatives as propositions or as sets of possible worlds they will form a Boolean algebra, provided they are closed under the operations mentioned above.

In the chapter on group decision processes I will study some assignment problems. The goal in these cases is to assign the members of one set to the members of another. The alternatives will be the different possible assignments. More rigorously, the alternatives can be thought of as the set of bijections between two equinumerous sets. This added structure gives access to, among other things, the well-known economic criterion of Pareto optimality.

These examples illustrate some possibilities of adding some structure to the set of alternatives. Within many fields of decision theory this added information is of major importance. Using axioms which characterize these structures along with some often simple axioms concerning preference orderings of the alternatives one can prove many strong and beautiful representation theorems.

4.

We next turn our attention to the decision situations. Generally speaking decision situations contain the relevant information on the 'values' of the outcomes. I will in this section introduce the distinction between individual and group decision-making and also briefly present some ways of representing individual 'values'. The term 'value' will be used here in a very broad sense and will include notions such as 'utility', 'desirability', 'strength of preference' and other similar expressions.

So far I have been speaking about the 'decision maker' although it is obvious that in many decision problems it is a

group rather than a single individual whose desires are considered. Luce and Raiffa have these words on the subject:¹⁰

"The distinction between an individual and a group is not a biological - social one but simply a functional one. Any decision maker - a single human being or an organization - which can be thought of having a unitary interest motivating its decisions can be treated as an individual in the theory. Any collection of such individuals having conflicting interests which must be resolved, either in open conflict or by compromise, will be considered to be a group.... Depending upon one's viewpoint, an industrial organization may be considered as an individual in conflict with other similar organizations or as a group composed of competing departments."

To this I would like to add that a decision made by a single person can sometimes (not to say often) be regarded as a 'group' decision. The members of the individual 'group' can be called the 'instincts' or, to use a less committing term, the aspects of the outcomes. A person may evaluate the alternatives differently according to the economic consequences, the pleasure the outcomes will give him, how they will affect his career or what people will think of him. It seems reasonable that the overall 'value' of the alternatives will be affected by the 'values' of all these aspects. I will take up this issue in a chapter on group decision processes (chapter III).

An important factor when classifying decision processes is how the individual values connected with the outcomes are represented in the situations. There are two major ways of representation: by preference orders and by utility measures. These two ways are of course not independent of each other; using (one-dimensional) utility measures you can easily form preference orders, and under certain conditions on a preference order there exists a utility measure which 'reflects' the preference order. Rather if a distinction is to be drawn it is whether you are willing to assume cardinal values (i.e. values in some sense related to numerical magnitudes) or only ordinal values (i.e. values containing information only on order).

Remember that in decision under certainty the alternatives can be identified with the outcomes. In the certainty case one has a direct evaluation of the alternatives, in other cases the values of the alternatives are often determined indirectly

by the value of the outcomes, e.g. by using expected utility.

A preference ordering is ordinarily a binary relation $\underline{P}$ which reports upon the 'values' of the outcomes (or the alternatives) only to the extent that ' $x \underline{P} y$ ' holds if the value of x is 'greater' than the value of y or, more simply expressed, if x is 'better' than y . In addition to the relation $\underline{P}$, a relation $\underline{I}$ of indifference is often also introduced with the intended meaning 'has the same value as'. For technical convenience also a relation $\underline{R}$ is introduced by the definition ' $x \underline{R} y$ ' if ' $x \underline{P} y$ ' or ' $x \underline{I} y$ '.

As regards the logical properties of these relations, there are conflicting intuitions as to which are the proper axioms for a preference relation. One should not be discouraged by such conflicts, but rather look upon them as grist for the mill of logical analysis. Indeed the study of logics for preference relations has become fashionable during the last two decades.¹¹ In spite of conflicts there are some fundamental conditions on these relations which have with some consensus been called "conditions of sanity". I present these as a series of axioms under the assumption that the reader is familiar with the basic terminology of relational properties.

Axiom 1: $\underline{P}$ and $\underline{I}$ are disjoint relations.

Axiom 2: $\underline{P}$ is asymmetric.

Axiom 3: $\underline{I}$ is symmetric.

Axiom 4: $\underline{I}$ is reflexive.

Axiom 5: $\underline{P}$ is transitive.

There has been some doubt concerning axiom 5. In experimental situations in which a subject is asked to choose the preferred outcome from several pairs of outcomes, it frequently happens that the resulting ordering is intransitive. Savage comments on this difficulty as follows:¹²

".... when it is explicitly brought to my attention that I have shown a preference for $\underline{f}$ as compared with $\underline{g}$, for $\underline{g}$ as compared with $\underline{h}$ and for $\underline{h}$ as compared with $\underline{f}$, I feel uncomfortable in much the same way that I do when it is brought to my attention that some of my beliefs are logically contradictory. Whenever I examine such a triple of preferences on my own part, I find that it is not at all difficult to reverse one of them. In fact, I find on contemplating the three alleged preferences side by side that at least one among them is not a preference at all, at any rate not any more."

If we add to the axioms above the condition that, for any alternatives x and y , either xPy , xIy or yPx , we obtain what is called a strict partial order. If we furthermore add as an axiom that I is transitive, we get a weak order. There are some other sets of axioms that have been proposed for preference relations, but these examples are sufficient for present purposes.¹³

A utility measure is a numerical measure of the values of the outcomes. As with any notion of quantity, it is necessary to specify the kind of scale on which a utility measurement is performed. In the classical representation theorems by von Neumann and Morgenstern /62/ and Herstein and Milnor /32/ the agent's preference ordering of the alternatives (which form a mixture space) determines the utility measure up to a linear transformation with positive coefficient, i.e. utility can be measured on an interval scale. In Jeffrey /34/, where the agent orders the elements of a Boolean algebra, the set of preference-preserving transformations of the utility function is larger, implying that utility can be measured only on a weaker scale.¹⁴ In any event, there is no basis for assuming that utility can be measured on scales stronger than interval scales.

What utility is supposed to measure was first stated by Bentham / 2/ as follows:¹⁵

"By utility is meant that property in any object, whereby it tends to produce benefit, advantage, pleasure, good, or happiness (all this in the present case comes to the same thing), or (what comes again to the same thing) to prevent the happening of mischief, pain, evil or unhappiness to the party whose interest is considered: if that party be the community in general, then the happiness of the community; if a particular individual, then the happiness of that individual."

Utility measures are of particular interest when dealing with individual decisions under risk. The utilities of the alternatives (and thereby also the preference ordering and the choice function) are computed from the utilities of the outcomes and the probabilities of the states of the world according to the so-called expected utility hypothesis.

I will return to topics of preferences and expected utilities in a chapter on qualitative probability and utility (chapter IV).

Preference orderings and utility measures are the main means of representing the values of the outcomes. There are, however, also other indirect ways. In assignment problems where the outcomes (alternatives) are bijections which assign to each member of a set an object from another set, it is often assumed that members evaluate objects, not bijections. The decision problem then becomes a question of determining the 'values' of the alternatives, i.e. the bijections, given information about the value of the objects.

The situation thus contains information on who the relevant decision makers are and for each of them (possibly only one) some kind of information as to the values attributed to each of the outcomes. It need not be true that all outcomes (and therefore possibly not all alternatives) are feasible in every situation. In an election for example, the feasible alternatives may be only those candidates having qualified in some previous selection, although many candidates are potential. In general, the set of feasible alternatives cannot be fixed in advance. Therefore the situation determines also which alternatives are feasible. One must then add the obvious restriction on the choice functions that the alternatives be chosen in a given situation from among the feasible ones.

III. GROUP DECISION PROCESSES

1.

A rational group decision can be viewed as a choice of the alternative (or alternatives), among a given set of alternatives, which has the greatest social value; where the social value of an alternative is obtained by some kind of aggregation of the individual values. For the most well-known group decision problem, the problem of how to find a 'fair' voting method, Arrow initiated the approach of studying how to aggregate preference relations of many individuals into a common social preference ordering. Arrow's approach differ from the one taken in the previous chapter to the extent that a choice function is indirectly defined, but obviously the choice of the group can easily be determined from the social ordering.¹⁶

This chapter will be devoted to a study of group decisions under certainty where individual values are represented by (weak) preference orderings. In section two I will argue that individual decision making often can be regarded as group decision making, where the 'group' consists of the individual's different aspects of the alternatives. I believe that such an interpretation has received insufficient attention. In section three I discuss the possibility of using utility measures rather than mere preference orderings to represent individual values in group decision making. I prove a theorem which shows that if one wishes to avoid intrapersonal comparisons of utilities, one cannot use more information in the method of aggregation than the information given by the preference orderings.

Section four gives a brief presentation of the theory of voting as it had developed since Arrow's pioneering results. I introduce a distinction between the positionalistic and the non-positionalistic approach to the evaluation of voting methods. A summary of the paper /PVF/ is then given.

The final section is devoted to a discussion of other group decision problems of importance for economists and political scientists. Particular forms of assignment problems are presented together with a summary of /AP/ and /MM/.

A word is in order on the nature of the choice functions used when group decision situations are involved. Previously, a choice function was defined simply as a function from the set of situations to subsets of the alternatives. For group decision choice functions it is necessary to specify further and in a stricter sense what will count as the set of situations. There are two main distinctions to be made: 1) whether the set of individuals in the group is a fixed set or if it is allowed to vary among different situations; and 2) whether all alternatives are feasible in all situations or whether only a subset of them may be included in the choice set.

As regards the first distinction I will assume, along with the major part of the literature on social choice theory, that the group of individuals is fixed and that for each of them there is in any situation a preference ordering of the set of alternatives. Recently, more attention has been drawn to choice functions which are defined for situations with a varying number of individuals and it has been shown that some surprisingly simple conditions on such functions lead to strong representation theorems.¹⁷

As for the second distinction I will discuss the question of feasible alternatives in the theory of voting in connection with Arrow's independence condition. For assignment problems, however, I will assume that all alternatives are feasible in any situation.

2.

This section is based on the hypothesis that individual decisions are always made according to some preference ordering of the set of alternatives, in the sense that the individuals always chooses the highest ranked alternative among the set of feasible alternatives. This assumption is not without implications for a descriptive theory of decision and my arguments will be balancing between the normative and descriptive views of the decision problems.

A preference relation was introduced as a binary relation P on the set of alternatives. We will now use the relation I

of indifference as defined from $\underline{P}$ by ' $x\underline{I}y$ ' iff not ' $x\underline{P}y$ ' and not ' $y\underline{P}x$ '. I will also use the relation $\underline{R}$ which has been previously defined as ' $x\underline{R}y$ ' iff ' $x\underline{P}y$ ' or ' $x\underline{I}y$ '.¹⁸

A hypothesis, which is popular among economists and which is the basis of classical utility theory, is that an individual preference ordering can be obtained directly from some utility measure on the set of alternatives. More explicitly, a function $\underline{u}$ is assumed which assigns numbers to the alternatives and a relation $\underline{P}$ is introduced by the definition

$$(1) \ x\underline{P}y \text{ iff } \underline{u}(x) \geq \underline{u}(y)$$

Such a definition endows the preference relation with certain properties. For example it follows immediately that

- (2) $\underline{P}$ is transitive and asymmetric,
- (3) $\underline{I}$ is an equivalence relation,
- (4) $\underline{R}$ is transitive and connected.

Comparing these properties with the results of experiments carried out by psychologists on human preferences one discovers several discrepancies. One of the most serious concerns the transitivity of the indifference relation. Indeed, in a great number of articles on preferences and decisions, both normative and descriptive, it is argued that the assumption of the transitivity of $\underline{I}$ is far too strong for a general theory of preference orderings.¹⁹

The definition (1) has therefore been modified to the following:²⁰

(5) $x\underline{P}y$ iff $\underline{u}(x) > \underline{u}(y) + \epsilon$, where ϵ is a (small) number. The interpretation of ' $x\underline{P}y$ ' is that x is definitely better than y . The number ϵ is a measure of what in psychology is called the 'just noticeable difference'.

A relation which can be described as in (5) is called a semicorder. It is characterized by the following properties:

- i) not $x\underline{P}x$,
- ii) if $x\underline{P}y$ and $z\underline{P}w$, then $x\underline{P}w$ or $z\underline{P}y$,
- iii) if $x\underline{P}y$ and $y\underline{P}z$, then $x\underline{P}w$ or $w\underline{P}z$.

It follows from these axioms that $\underline{P}$ is a transitive relation, but it does not follow that $\underline{I}$ is a transitive relation.

I will next turn to another extension of equation (1), where each alternative is measured in several 'dimensions' and where the preference ordering is based on some kind of aggregation of the values. It is reasonable to suppose that a decision maker regards the alternatives from different aspects and highly unreasonable to suppose that the 'values' of these aspects or factors can be measured on a common scale. For example, imagine a doctor who chooses among several medicines for a patient. Here the relevant aspects are the prices and the effectiveness of the alternative medicines, aspects which cannot be measured on a common scale without unnatural assumptions. In this case it seems defensible to employ the decision principle that the medicines should be chosen primarily according to their effectiveness and only in the second place, when two medicines are equally efficient, according to their prices.

Now, from a more psychological point of view, individual decision making can also be looked upon as a 'group' decision. An individual often has to make up his mind which act to perform in a given situation. The acts from among which he chooses may as well be called the alternatives. These alternatives may be valued differently according to the aspects under consideration. For example, the individual may be interested in the economic consequences of the acts, in the degree to which they satisfy his desires or in the way people will think of him if he performs a certain act. It is plausible that he will evaluate the acts according to some aggregation of these factors.

If we assume that for a certain set of decision problems there exists a set of 'relevant' factors or at least a set of 'dominating' factors, then the decision is a group decision in the sense that the different factors are the 'individuals' of the group. In other words, the evaluation of different factors by a single individual is structurally equivalent to the forming of a group decision. This approach is at least consistent with several modern psychological theories of personality.²¹

I will now briefly present some commonplace decision principles for individual decision making when alternatives are evaluated from several aspects. Assume there is a fixed set of

n factors and, initially, that a number can be given as the value for each factor of each alternative. As a consequence, each alternative can be represented by an n-dimensional vector, one dimension for each factor. As earlier indicated, I will not directly consider a choice of a best alternative but rather a preference ordering of alternatives.

A very straightforward way to evaluate the alternatives is to find some kind of average of the numbers assigned to the factors of each alternative. The simplest way is to take their arithmetic mean or, since each alternative has the same number of factors, simply add the numbers assigned to the factors. This leads to the following definition of the order relation $\underline{P}$:

$$(6) \quad (x_1, \dots, x_n) \underline{P} (y_1, \dots, y_n) \text{ iff } \sum_i x_i > \sum_i y_i.$$

This definition can be viewed as a slight variation of classical utility theory, since the alternatives are ordered according to an overall utility measure as determined by the sum of the utilities of the factors (or, equivalently, by their arithmetic mean). There exist of course other ways of finding an 'average' of the values of the components of an alternative, but, as far as I know, little consideration has been given them in the literature.

The maximin principle, which has been suggested as a principle for decision making under uncertainty, can also be applied to the present problem. Of two alternatives, the one with the smallest number for any factor is the worst. This principle is defensible in some situations. Imagine for example that a government at war must choose among alternatives each of which will cause losses in several cities. Then one might argue that the best alternative is that for which the greatest loss for any city is as small as possible, a decision which follows from the maximin principle.

Both the method based on additive utility and the maximin principle are cardinal in the way that their results are dependent on the specification of the actual numerical values of the different factors. I will now describe two methods which are ordinal in the sense that their results do not depend on exact numerical values, but merely on how the numbers are related to each other in terms of greater than, less than and equal to.

In the previous example of the doctor, one of the factors, efficiency, dominated the other, prices, in the determination of the ordering of the alternatives. This principles can be extended to an arbitrary number of factors by assuming an ordering of the importance of the factors. The ordering of the alternatives is then in the first place based on the magnitudes of the values of the most important factor and if two alternatives have the same value for the most important factor then the second most important determines their ordering. If the values are the same also for the second factor, then the third most important is crucial etc. This method has been called the lexicographic ordering method.

It is interesting to consider whether, by assigning very large numbers to the important factors and small ones to the less important, one can represent the lexicographic method as a special case of classical utility theory with additive utility. It turns out that for a fixed finite set of alternatives, any lexicographic ordering can be represented as an addition of utilities in this way, but it is easy to construct counterexamples when the set of alternatives is infinite.²²

The interpretation of individual decision making as an aggregate of several factors has been studied in an article by Plott, Little and Parks /49/. They assume that all factors, or properties as they call them, are purely ordinal. They are not interested in any utility measurement of the factors, but only in orderings of the alternatives according to the different properties. They introduce conditions for 'rational' aggregating methods, which in concept are similar to those proposed by Arrow /1/. Their main result is that if the conditions hold, individual choice is necessarily 'dictated' by one property alone, i.e. the decision is made according to some lexicographic principle.

In experimental settings where the subject is asked to make pairwise comparisons of the alternatives an intransitive ordering is often observed. One way to explain such a seemingly 'irrational' preference ordering, an explanation which seems to be overlooked,²³ is to assume that binary preferences are based on the alternative which dominates with respect to most factors.

This 'decision principle' is then a sort of majority decision, and it is well-known that intransitivities can occur in majority decisions.

There are of course several other important decision principles, both cardinal and ordinal. My main aim has been to show how individual decision making can be treated under the heading 'group decision theory', an issue which seems to have been neglected so far.

3.

In the next section on the theory of voting I will work with the assumption that individual values are represented by preference orders, rather than by a utility measure according to which the alternatives are ordered. As a preliminary to this approach the present section examines the consequences of assuming the existence of utility measures.

As with any kind of measurement, the question of relevant scales is important. We have seen that the representation theorems in the literature seem to suggest that utility is at best measureable on an interval scale, i.e. the utility function can be uniquely determined up to a positive linear transformation.²⁴

Since more information can be obtained from a utility measurement of the alternatives than from a preference ordering, it might be thought that this extra information can be used when aggregating the individual values of the alternatives. The main problem is that even if every individual in the group can represent his values by a utility measure, one cannot conclude that the same scale can serve for all the measurements. This is the well-known problem of interpersonal utility comparisons, which has been much debated in the economic, psychological and philosophical literature. I will take the standpoint here, along with most writers on the subject, that the utility concept is still too vague to allow any empirically meaningful interpersonal comparisons.²⁵ Instead, my aim is to study the possibility of finding a method of aggregating utilities which is independent of the scales chosen for each individual.

Preference orderings can be viewed as measurements of values

on an ordinal scale, i.e. a measurement where the numbers assigned to the alternatives are unique only up to a transformation with a monotone increasing function.²⁶ The problem of finding an aggregation method which uses the extra information given by a utility measure is therefore reducible to the question of whether any method exists such that the outcome is invariant under positive linear transformations of the utilities, but not under all monotone increasing transformations.

In his monograph, Arrow mentions the problem as follows:²⁷

"Even if, for some reason, we should admit the measurability of utility for an individual, there still remains the question of aggregating the individual utilities. At best, it is contended that, for an individual, his utility function is uniquely determined up to a linear transformation; we must still choose on the infinite family of indicators to represent the individual, and the values of the aggregate (say a sum) are dependent on how the choice is made for each individual. In general, there seems to be no method intrinsic to utility measurement which will make the choices compatible. It requires a definite value judgement not derivable from individual sensations to make the utilities of different individuals dimensionally compatible and still a further value judgement to aggregate them according to any particular mathematical formula. If we look away from the mathematical aspects of the matter, it seems to make no sense to add the utility of one individual, a psychic magnitude in his mind, with the utility of another individual."

I will now proceed to prove more formally that there is "no method intrinsic to utility measurement which will make the choices compatible".

If n is the number of individuals, the alternatives can be represented as n -dimensional vectors of real numbers where the i th component stands for the i th individual's 'value'. Since I am assuming that the scales can be changed by positive linear transformations, the vector $(x_1, \dots, a \cdot x_i + b, \dots, x_n)$ will be in the relevant set of alternatives as soon as $(x_1, \dots, x_i, \dots, x_n)$ is. Since b can be any real number, the set of alternatives will be the n -dimensional vector space of real numbers.

We are looking for a method which will choose the 'best' alternatives. This choice can be represented by a binary relation R and the assumption that the alternative x is among the chosen if and only if not $y R x$ for any other alternative y . R can be thought of as a rudimentary preference relation, since I

do not want to impose any structural conditions on the relation.

Theorem: Let R be any binary relation, defined on the n -dimensional real space, with the following property:

- (1) $(x_1, \dots, x_i, \dots, x_n) R (y_1, \dots, y_i, \dots, y_n)$ if and only if $(x_1, \dots, a \cdot x_i + b, \dots, x_n) R (y_1, \dots, a \cdot y_i + b, \dots, y_n)$ for all real numbers a and b such that $a > 0$.

Then the following holds:

- (2) $(x_1, \dots, x_i, \dots, x_n) R (y_1, \dots, y_i, \dots, y_n)$ if and only if $(x_1, \dots, f(x_i), \dots, x_n) R (y_1, \dots, f(y_i), \dots, y_n)$ for every monotone increasing function f .

Proof: I will give the proof in some detail, although the idea is simply to study the line through $f(x_i)$ and $f(y_i)$. Let f be an arbitrary monotone increasing function and let $(x_1, \dots, x_i, \dots, x_n)$ and $(y_1, \dots, y_i, \dots, y_n)$ be two arbitrary vectors such that $(x_1, \dots, x_i, \dots, x_n) R (y_1, \dots, y_i, \dots, y_n)$.

i) Suppose $x_i = y_i$. We then have $f(x_i) = f(y_i)$. If we choose $a = 1$ and $b = f(x_i) - x_i$ we get $a \cdot x_i + b = f(x_i)$ and (2) follows immediately from (1) since the vectors now are identical.

ii) Now suppose $x_i > y_i$. Since f is monotone and increasing, we have $f(x_i) > f(y_i)$. Choose $a = (f(x_i) - f(y_i)) / (x_i - y_i)$ and $b = (x_i \cdot f(y_i) - y_i \cdot f(x_i)) / (x_i - y_i)$. Then we get $a > 0$, $a \cdot x_i + b = f(x_i)$ and $a \cdot y_i + b = f(y_i)$ and (2) follows from (1) as before.

iii) The case when $x_i < y_i$ is proved in the same way as case ii) and the proof is complete.

Note that both multiplication by a positive constant and addition of a constant are needed in the proof. If we only demand R to be invariant under multiplication with a positive constant of the components of the vector, then the relation R' defined by $(x_1, \dots, x_i, \dots, x_n) R' (y_1, \dots, y_i, \dots, y_n)$ iff $\prod_i x_i \geq \prod_i y_i$ will satisfy this requirement but not (2). Similarly, the relation R'' defined by $(x_1, \dots, x_i, \dots, x_n) R'' (y_1, \dots, y_i, \dots, y_n)$ iff $\sum_i x_i \geq \sum_i y_i$ will be invariant under addition of a constant to the components, but R'' does not satisfy (2).

This theorem supports Arrow's view that if one wants to avoid interpersonal comparisons of utility, then one cannot use the extra information given by the utility measure, but might as well start from the much weaker assumption that individual values

are represented by preference orderings.

Note that if one has a finite set of alternatives with vectors of individual utilities to represent them, one can always impose the choice of a scale by some normalization procedure. E.g. one can transform the personal scales so that the best alternative for each alternative for each individual will have utility 1 and the worst utility 0. When the scales are fixed in this ad hoc manner one can determine the social ordering of the alternatives simply by summing the utilities. This provides a social ordering of the alternatives which is invariant under linear transformations of the original utilities. The main disadvantage of this method, and the reason why it is irrational, is that the social preference relation between two alternatives can be changed by changing the utility of a third alternative for some person. The requirement that any method of aggregating be independent of such changes resembles of Arrow's independence condition, but is in fact a much weaker condition. I will return to the topic of independence conditions in the next section.

4.

We will in this section concentrate on a particular group decision problem — the problem of voting. As a theoretical issue the problem has a fairly long history, with the classical highlights being the papers by Condorcet, Borda and Dodgeson.²⁸ Nevertheless it was not until the appearance of Arrow's monograph /1/ in 1951 that interest in the theory of social choice became general. Since then the theory has had a rapid development, mainly due to the many paradoxes that have been shown to follow from seemingly reasonable assumptions.

Usually, the set of alternatives is assumed to be a finite set without any added structure, and I will throughout this section operate under this assumption. I will also assume that there is a fixed finite group of voters (or individuals) who are all familiar with the alternatives. Preference orders will be assumed to represent individual values. More precisely, we assume that for each individual i in a given decision situation a preferences are represented by an ordering R_{ia} which is a weak order, i.e.

connected and transitive.

A voter will be allowed to choose any weak ordering of the alternatives as his preference relation, so there are no a priori restrictions on which preferences an individual may have. In certain contexts this is a rather unnatural assumption; in parliamentary decision situations one can often recognize an ordering of the alternatives on a scale from left wing radical to right wing conservative. In such circumstances it might be thought irrational for the individual to rank a leftist alternative first, a rightist alternative next and last an alternative from the center. But the apparent irrationality follows from an imposed structure of the alternatives from left to right. Since I want to think of the alternatives as forming an unstructured set, I shall permit every conceivable weak ordering of the alternatives in the definition of the range of the decision function. For an extensive treatment of the theory of voting when preference orderings are 'single-peaked', i.e. based on a scale from left to right, the reader is referred to Black /3/.

A situation will be defined as an n -tuple of preference orders, one for each individual. The decision function will be a function from situations to weak orderings of the alternatives. Such a decision function will be called a voting function. My definition of the decision function entails that all alternatives are regarded as feasible in every situation. This is essentially the approach taken by Arrow in /1/.

There are several other ways of defining the decision function for the voting problem, one of the most important being that advocated by Plott and Fishburn.²⁹ In their approach the situation is conceived of as a set of preference orders together with information on which alternatives are feasible. The decision function is a choice function which picks out, for a given situation, a non-empty subset of the feasible alternatives.

A decision function which gives as its result a preference ordering of the alternatives is less general than a choice function since there is a natural way of defining choice functions in terms of preference orders, but not vice versa. It might therefore be thought that Arrow's paradox and similar results can be avoided in the more general framework, but, as Bengt Hans-

son has shown in several places,³⁰ theorems about voting functions in Arrow's decision framework can very often be proven with respect to choice functions.

One of the interesting features of Arrow's work is the introduction of a set of conditions on voting functions which seem reasonable. He then searches for voting functions which satisfy these conditions and, quite disturbingly, finds none. Arrow's starting with conditions contrasts with most earlier writers on the topic who proposed tentative voting methods and then argued for or against them, using particular situations as examples and counterexamples.

I will now briefly discuss some of Arrow's conditions along with some others which have been proposed for voting functions. I will give neither an exhaustive list nor precise definitions of the conditions. For the interested reader, the major conditions in this area are discussed and precisely defined in /PVF/.

The surprising fact about Arrow's seemingly reasonable conditions is that they are inconsistent, or, more precisely, that there is no voting function which satisfies them all. Most writers blame this inconsistency on the condition on independence of irrelevant alternatives, which may be defined somewhat loosely in the following way:

Definition 4.1: Suppose the preference orderings for each individual in situations a and b coincide as regards the alternatives in a subset B of all the alternatives. Then a voting function satisfies the condition of independence of irrelevant alternatives iff the social orderings (i.e. the outcomes of the voting function) in situations a and b coincide as regards the alternatives in B. An example is given to illustrate the effects of this condition. Compare the following two voting situations where there are three alternatives and three voters:

<u>a</u> :	1.	2.	3.	<u>b</u> :	1.	2.	3.
	x	y	y		x	y	z
	y	z	z		z	x	y
	z	x	x		y	z	x

Since the individual preference relations regarding the alternatives x and y are the same in both situations, the independence condition demands that the social ordering of these alterna-

tives should be the same in both situations. So, if x is socially preferred to y in a it is also preferred to y in b and vice versa.

The well-known economic criterion called Pareto optimality was restated by Arrow as follows:

Definition 4.2: A voting function satisfies the condition of Pareto optimality iff whenever every individual prefers an alternative x to an alternative y , then x is socially preferred to y .

Arrow showed that the independence condition, Pareto optimality and a third condition, that there be no voter who dictates the decision of the group, are inconsistent in his framework. They are inconsistent because no voting function, as he had defined the concept, satisfies all three conditions. His proof will not be repeated here since it can be found in several other publications.³¹

The condition that there be no dictator can be strengthened as follows:

Definition 4.3: A voting function satisfies the condition of symmetry iff every voter has the same influence on the decision.

This condition is sometimes referred to as the egalitarian principle.

In many parliamentary voting procedures the alternative 'maintain the status quo' holds an exceptional position. In order to avoid this bias, the following condition was introduced.

Definition 4.4: A voting function satisfies the condition of neutrality iff no alternative is favoured for other reasons than the preferences expressed in the individual preference orders.

Bengt Hansson proved in /25/ that there is only one voting function which fulfils the conditions of independence, symmetry and neutrality, to wit, the degenerate function whose social ordering in every situation is the preference relation where all alternatives are considered equally good. Of course, this voting function is of no importance.

Several modifications of Arrow's theorem have been made. For example Bengt Hansson showed that Arrow's proof can be extended to the case where choice functions are used instead of

voting functions. For other examples, the reader is referred to Fishburn /18/, Murakami /46/ and Sen /57/.

Although the independence condition has recently been of prime interest in the theory of voting, historically it is the majority principle which has provoked the most debate. Most parliamentary voting procedures are based on the majority condition. The ascendancy of the majority principle seem to arise out of the belief among politicians and political scientists that voting functions which satisfy the majority principle are the most natural extensions of the majority rule for two alternatives. The principle has also been called the Condorcet criterion after the first person to study it theoretically.

Definition 4.5: A voting function satisfies the majority principle iff any alternative which is preferred by a majority of the voters to each of the other alternatives is also an alternative which is preferred to all other alternatives in the social preference order.

The question arises whether the majority principle and the condition of independence of irrelevant alternatives are consistent; Arrow believed them to be so in the addition written for the second edition of his book³² and few others have expressed any doubt as to their consistency. However, Bengt Hansson and I offer a simple proof, which the reader will find in /PVF/, which establishes that these two conditions are indeed inconsistent when there are at least three voters and three alternatives. In fact an independence condition weaker than Arrow's is sufficient to give the inconsistency³³.

The independence conditions and the majority principle are based on the view that the social preference relation between a pair of alternatives should depend only on the individual preference relations between these alternatives regardless of their general positions in the individual orderings. I call this approach the 'non-positionalist view' of the theory of voting which contrasts with the 'positionalist' view in which the positions of the alternatives in the individual preference orders are given a crucial role in determining the social ordering.

If one wants to give a strong interpretation of the above-mentioned inconsistency between the independence condition and

the majority principle, one can say that the non-positionalist view is inconsistent in itself. This is a good argument for turning to a closer study of the positionalist approach to the theory of voting, the main aim of the paper /PVF/.

Positionalist voting functions have been neglected by most theorists. Only the Borda function, which is the most well-known positionalist voting function, is used as an example to contrast with functions based on the majority principle.

Typical positionalist voting functions are those where numbers are assigned to the alternatives according to their positions in the individual preference orderings and where the social ordering then is obtained from the sum of these numbers for each alternative. Voting functions which can be described in this manner are called representable in /PVF/. The numbers assigned to the alternatives are, in a sense, imposed by the voting function in question and are by no means used as a kind of utility measure for the different individuals.

A good example of a representable voting function is Borda's function. Borda's original definition of the function was intended for linear preference orders only, i.e. orderings without ties. He proposed that the least preferred alternative in each individual's preference order should be assigned the number 0, the next to the last 1, the next again 2, et.c. This definition can be extended to weak orders if one adds the provision that each alternative of a tie is assigned the mean value of what they would have been assigned if they were ranked in a linear order.

Borda defended his method in terms of utility measure, although it is possible to describe the method without the concept of utility. Borda's arguments for his function have often been misinterpreted as showing that the function is actually an odd way of aggregating utilities.³⁴

As an example of how Borda's procedure works, take the following situation:

<u>a</u> :	1.	2.	3.
	x	z	y
	z	u	x
	y	xy	z
	u		u

In the first preference order x is assigned 3 points, z , 2 points, y , 1 point, and u , 0. In the second z is assigned 3 points, u , 2 points, and x and y 0.5 points each. In the third y is assigned 3 points, x , 2, z , 1, and u , 0. The alternative x is in total assigned 5.5 points, y is assigned 4.5, z is assigned 6 and u only 2 points. So the social preference ordering in this situation has z as the best alternative, x as the second, then y and finally u .

The properties of the Borda function are studied to some extent in /PVF/. A set of conditions is given which uniquely characterizes the Borda function among the set of representable voting functions. The most important condition is a stability condition which in essence demands that if some individual changes his preferences slightly, then there will be no radical change in the social ordering.

The question naturally arises whether there are any representable voting functions which satisfy the majority principle. Not too surprisingly, it can be shown that if we have more than three alternatives or more than three voters, then there is no representable voting function which satisfies the majority principle. Similarly it follows immediately from one of Hansson's theorems³⁵ that no neutral representable voting function satisfies Arrow's independence condition. Thus, there is a clear boundary between representable voting functions and the non-position-alistic conditions.

Representable voting functions satisfying neutrality can be characterized by the property that alternatives in preference orders with the same topological structure will be assigned the same numbers. Similarly, a representable voting function satisfies Pareto optimality if and only if the number assigned to an alternative x is greater than the number assigned to y iff xRy . Finally, one may note that all representable voting functions satisfy symmetry by definition.

The theorems in /PVF/ show that representable voting functions in general have desirable properties. The main criticism of them and other positionalist voting functions has been that they do not satisfy the majority principle. However, this is only a matter of taste, you either like positionalism or don't.

5.

The main interest in group decision theory has been the problem of finding an optimal voting method. There are however, other problems which involve some kind of group decision, for example the problem of optimal staff distribution and estate distribution. The aim of this section is to present informally these and some other related group decision problems and to discuss how these problems can be analyzed in a way similar to that of the theory of voting.³⁶ Also a summary of the papers /AP/ and /MM/ will be given.

The problem of optimal staff distribution, one form of assignment problem, can be described as follows. Suppose there is a set of employers and a number of persons looking for employment. The employers are assumed to have preferences among the potential employees, i.e. they regard some of the persons looking for employment as more suitable for the job than others. The problem is then to distribute the jobs in an 'optimal' way.

This problem has been solved using standard techniques from linear programming under the assumption that a person's adaptability for a certain job can be measured on a common scale. But, as indicated in section 3, as I wish to avoid the problem of interpersonal comparisons of utilities, which is the weight of this assumption, I will allow only an ordering of the persons according to their suitability for a certain task. In /AP/ this assignment problem is regarded as a problem of social choice, where preferences orderings are used as the only initial information.

A decision function for this problem will be called an assignment function. It takes situations, which again are n-tuples of preference orderings (one for each job, ordering the persons), as arguments and the value of an assignment function will be a one-one relation between the persons looking for employment

and the jobs. Such a one-one relation will be called an assignment. Of course the set of vacant jobs and the set of persons looking for employment may initially not be equinumerous, but any discrepancy can easily be eliminated by adding dummy jobs or persons. In general, I will allow the assignment function to pick out a non-empty subset of all possible assignments, a subset because there may be no reason to distinguish two or more equally good assignments.

In /AP/ several conditions for assignment functions are presented. Some of them borrow their content and ethical justification from the theory of voting, but most of them are unique for this problem. One of the most important conditions is closely related to classical Pareto optimality. In the context of the staff distribution problem, this condition demands that there be no reassignment of persons and jobs such that at least one job is filled by a better person while no job is filled by a less qualified person. One can formulate counterparts to the symmetry and neutrality conditions of the previous section for the assignment problem.

A condition which in concept is an independence condition is also introduced, but it easily shown that this condition along with some other plausible conditions lead to inconsistencies. Since the arguments in favour of an independence condition in this context seem to be weaker than for the voting problem, the attempt to include such a condition is abandoned.

In the paper I suggest several assignment functions and examine them to see which of the proposed conditions are fulfilled. One of the functions picks out the set of all Pareto optimal assignments in every situation. It is shown by a simple example that this set is intuitively too large and that further conditions are needed. Also counterparts to representable voting functions are introduced and, again, it turns out that these 'positionalistic' functions have desirable properties.

In /MM/ I take up a problem, which can be viewed as a generalization of the assignment problem. This problem has been called 'the stable marriage problem' since one interpretation (naive, but simple to describe) is to find an 'optimal' set of marriages between the members of a set of women and a set of men, given

that each of the members of a set has preferences among the members of the other set.

Formally, we have two disjoint sets which are assumed to be equinumerous (if necessary through dummy items). In this case, situations differ from previous types in that they consist of two parts; in the first part each member of the first set has a preference ordering of the members of the second set and in the second part the members of the second set order the members of the first according to their preferences. The decision of a 'two-way' assignment function is the same as for the assignment problem previously described, viz. a one-one relation between the two sets (or possibly several such relations). Such a one-one relation is also here called an assignment.

There are many other interesting and potentially important applications of the formal problem. The staff distribution problem can be structurally related to the marriage problem if we not only allow employers to express their preferences among the persons seeking employment, but also give the presumptive employees the opportunity to express their opinion about the vacant jobs. If we replace the employers by a set of colleges and let the persons looking for jobs be students, then we have the problem of college admission. For this problem the theoretical analysis seems to have a direct application and a computer algorithm has already been written for a special case.³⁷

As regards conditions that seem reasonable for assignment functions of this type, there is one which has dominated in previous publications. This condition is called stability,³⁸ which in the marriage problem demands that any man and any woman who are not assigned to each other do not both prefer each other to their present partners. Gale and Shapley, in the original paper on this problem, proved that if preference orders are all linear orders, it is always possible to find at least one stable assignment in any situation. They also gave an algorithm to find such an assignment.

In /MM/ their results are extended to the case where preference orders are allowed to be weak orders. It is also shown that stability by itself is not sufficient in some situations to ensure that the assignments will conform to our intuitions about

what is an 'optimal' assignment. Stability in connection with another condition which I draw from the idea of Pareto optimality, provides more assurance.

Even if some structural conditions similar to symmetry and neutrality in the previous section are introduced, there are still several assignments in most situations which will satisfy these conditions. Therefore one might want to impose further conditions in order to narrow the class of 'admissible' assignments.

One way of doing this is to introduce a relation $\underline{M}$ on the assignments which holds between two assignments h and h' whenever a majority of the persons involved are better off under h than under h' . As its counterpart in the theory of voting, the relation $\underline{M}$ is not transitive. In some situations, however, it is possible to find an assignment which is preferred by a majority to all other assignments, in which case one can demand that this assignment should be selected. This selection in no way contradicts the previous conditions since it can be shown that any such assignment is stable and Pareto-optimal.

There are many other problems which are amenable to similar kinds of analysis. Take for example the estate distribution problem. An estate, consisting of a number of commodities, is to be divided in a fair manner among the heirs. If they can give a monetary evaluation of the items, a method can be found,³⁹ which works even if the heirs cannot compare their evaluations. If only preference orderings are allowed the problem becomes similar to the earlier assignment problems. The main difference is that one can no longer talk of an optimal one-one relation between the items and the heirs, since it may very well be that somebody gets more than one item as his fair share. Some other construction must be tried, but at the moment I shall leave the study of this problem as an open project.

Still other group decision problems are described in Gärdenfors /20/, in which preference orders and not utility measures seem to be the natural basic information. Since most of these problems have some economically interesting aspects I consider these problems to be well worth attacking.

IV, SUBJECTIVE PROBABILITY AND EXPECTED UTILITY

1.

I will in this chapter consider decision making under risk. The most important approach to individual decision making when the outcomes are uncertain is the theory of decision based on expected utility. For this theory it is assumed that there is a probability distribution $\underline{P}$ over the states of the world which represents the decision maker's subjective degrees of belief and a utility measure $\underline{u}$ which represents the personal utilities of the outcomes.⁴⁰ If one for the moment assumes that there are only finitely many possible states, one can define the expected utility of an alternative in the following manner. Let o_{xj} denote the outcome if alternative x is chosen and the true state is j and let p_j be the probability that j is the true state. Then the expected utility $\underline{u}(x)$ of the alternative x is defined by $\underline{u}(x) = \sum_j p_j \cdot \underline{u}(o_{xj})$. In the more general case with infinitely many states of the world, the sum can be replaced by an integral in an obvious manner.

Another way of introducing expected utility is to take Jeffrey's approach and identify the alternatives with subsets of the set $\underline{W}$ of possible states. In this case, the utility function $\underline{u}$ assigns numbers to the states instead of the outcomes. The expected utility $\underline{u}(A)$ of an arbitrary (non-empty) subset A of $\underline{W}$ is defined by $\underline{u}(A) = \sum_{j \in A} p_j \cdot \underline{u}(j) / P(A)$ where $\underline{P}$ is the given subjective probability measure and p_j is the same as $\underline{P}(\{j\})$. Here as well, the sum in the general case is replaced with an integral.

Either way, one can compute the utility of the alternatives. These numbers can then be used to define preference orderings and choice functions in a very straightforward manner. Also the probability measure over the set of states of the world naturally gives an ordering of all subsets of the set of states according to their probabilities. In standard terminology, the subsets of the set of states are called events.

A commonplace assumption about human behaviour is that the alternative chosen in a decision situation is the alternative

having the greatest expected utility. Though innocent-looking, this assumption rests on the postulate that for every individual there is both a subjective probability measure and a personal utility measure. Although it might be difficult to determine which probability and which utility measures form the basis of individuals' decisions, the existence assumption can in principle be tested by checking whether the choices and preferences among a set of alternatives have the properties consistent with a determination based on expected utility.

A natural problem is then to find axioms on the orderings of alternatives and events which guarantee the existence of probability and utility measures such that the orderings agree with the probabilities and expected utilities obtained from these measures. Depending on the structure assumed for the set of alternatives different axioms on the ordering of the alternatives are required to establish the existence of a utility measure. The ordering of events according to their degrees of probability is, however, independent of the structure of the alternatives.

I will first turn to axiom systems for subjective probability based on qualitative orderings of the events, i.e. the subsets of the set of states of the world. This will lead up to a presentation of /QP/. Thereafter I will briefly discuss axiomatic systems yielding theorems on the existence of utility measures, concentrating on the case where the set of alternatives form a Boolean algebra as in Jeffrey /34/.

2.

Suppose we have a decision maker in a decision situation comprising a fixed set of relevant states of the world, and therefore a fixed set of relevant events. Let the agent say for a number of pairs of events which of each pair he believes to be the more probable. The question of interest is whether there exists a probability measure consistent with the agent's binary comparisons. The goal is to find conditions on the relation 'is at least as probable as' which guarantee the existence of such a measure.

The first major step towards a solution is the set of conditions proposed by de Finetti /11/.⁴¹ Where A , B and C denote arbitrary subsets of the set $\underline{W}$ of states (i.e. A , B and C are events) and where the sign $\geq$ denotes the relation 'at least as probable as', one may write de Finetti's conditions as follows:

- 1) if $A \geq B$ and $B \geq C$, then $A \geq C$,
- 2) $A \geq B$ or $B \geq A$,
- 3) if $A \cap C = \emptyset$ and $B \cap C = \emptyset$, then $A \geq B$ iff $A \cup C \geq B \cup C$,
- 4) $A \geq \emptyset$
- 5) $W > \emptyset$ (i.e. not $\emptyset \geq W$)

These conditions are clearly necessary for any ordering of the events based on a given probability measure on $\underline{W}$. As it is easy to show that many of the expected properties of qualitative ordering of the events follow from de Finetti's axioms one might ask whether his axioms are sufficient to guarantee that there is always a probability measure $\underline{P}$ on $\underline{W}$ such that $\underline{P}(A) \geq \underline{P}(B)$ iff $A \geq B$. Where the set $\underline{W}$ has infinite cardinality they are not sufficient as shown by a counterexample in Savage /54/.⁴² But it remained an open question for a surprisingly long time whether the conditions are sufficient where $\underline{W}$ consists of finitely many states. De Finetti conjectured that they were, but he also added that if his conjecture was wrong, then it was because the 'right' axioms had not been found.

Kraft, Pratt and Seidenberg /37/ found an ingenious counterexample for a set $\underline{W}$ containing five elements. Call these elements x , y , z , u and v and consider the relations $\{z\} > \{x,y\}$, $\{y,u\} > \{x,z\}$ and $\{x,v\} > \{y,z\}$. For any probability measure $\underline{P}$ on $\underline{W}$ which agrees with these three relations it holds that $\underline{P}(\{x,y,z\}) \leq \underline{P}(\{u,v\})$. Kraft, Pratt and Seidenberg showed that the relation $\{x,y,z\} > \{u,v\}$ together with the three earlier did not violate de Finetti's axioms. They also gave necessary and sufficient conditions for an ordering of the subsets to agree with a probability measure, but the ideas behind their conditions were difficult to understand (partly due to their cumbersome notation). Fortunately a much more lucid pre-

sentation of their conditions and the proof that they are necessary and sufficient are given in Scott /55/. Scott's version of the conditions is stated here.

Theorem: Let S be a finite set and $\succeq$ an ordering of the subsets of S . Necessary and sufficient conditions that there exists a probability measure P such that $X \succeq Y$ iff $P(X) \geq P(Y)$ for any subset X and Y of S are that the following hold:

- 1) $X \succeq \emptyset$
- 2) $X \succeq Y$ or $Y \succeq X$
- 3) $S \succeq \emptyset$ (i.e. not $\emptyset \succeq S$)
- 4) for all natural numbers $m > 0$ and for all subsets $X_0, \dots, X_m, \dots, Y_m$ of S , if $X_i \succeq Y_i$ for $0 \leq i < m$ and any element of S belongs to exactly as many X_i 's as Y_i 's, then $Y_m \succeq X_m$.

To illustrate the power of the fourth condition, I show how it is violated by the Kraft-Pratt-Seidenberg counterexample to de Finetti's conditions. Every member of S belongs to exactly as many of the sets $\{x,y\}$, $\{x,z\}$, $\{y,z\}$ and $\{u,v\}$ as of $\{z\}$, $\{y,u\}$, $\{x,v\}$ and $\{x,y,z\}$, so if the first three relations among the previous stated hold, the fourth should not according to the fourth condition of the theorem.

The theorem above solves the problem of finding the appropriate conditions on an ordering of the events according to unconditional subjective probability measures. The problem of finding conditions on an ordering of conditional probabilities was first studied by Koopman /36/. He had as primitive a four-place relation $X/Y \succeq Z/W$ on subsets X, Y, Z and W of S with the intended meaning " X given Y is at least as probable as Z given W ". Koopman stated a set of conditions on this relation which are necessary (but not sufficient) if there is a probability measure P such that $X/Y \succeq Z/W$ iff $P(X/Y) \geq P(Z/W)$. By adding a partition condition, essentially similar to conditions used by de Finetti /11/ and Savage /54/, which forces the set S to be infinite, Koopman could show that these conditions were sufficient for the ordering to agree with a probability measure. The set of necessary and sufficient conditions when S is finite was found recently by Domotor /12/, who applied the general method

suggested in Scott /55/. Domotor's conditions are comparatively complicated, so we omit them.

In /QP/ it is shown how the theorems by Scott and Domotor can be used as a basis for an intensional logic for qualitative probability. This approach was first made in Segerberg /56/ and my paper is to a great extent a further development of his ideas. The paper's value lies not so much in extending our understanding of qualitative or subjective probability in the context of decision processes, but rather in stating a theory in a formal language and eliminating all irrelevant elements.

The formal language used is extremely simple; besides truth-functional connectives and propositional variables there is only one binary intensional operator ' $\geq$ ', which of course is intended to mean "is at least as probable as". It is shown how the conditions of Scott's theorem can be translated into this language, the hardest problem being the phrase "any element of S belongs to exactly as many X_i 's as Y_i 's". Scott formulated this phrase in terms of characteristic functions. Domotor pointed out how it could be translated into a system of identities among sets, which Segerberg then showed how to formulate in an intensional language. Together with truth-functional tautologies and an axiom of extensionality these translations of the conditions are the axioms of the basic logic for (unconditional) qualitative probability.

One of the advantages of using an intensional language is that one can deal with higher order probability statements, i.e. statements about probabilities of probabilities, without further ado. Whether such statements are meaningful is of course open to question, but certainly there is no harm in having the power to treat them in the same way as ordinary probability statements.

The base of a semantical model for the intensional language is a set $\underline{W}$ of worlds, where the worlds can also be thought of as different 'states'. Associated with each of the worlds is a probability measure over $\underline{W}$, not necessarily the same measure for all of the worlds.⁴³ Every atomic sentence in the language

is assigned a subset of $\underline{W}$, interpreted as the set of worlds where the sentence is true.

When evaluating the truth of compound sentences in a model of this kind, truth-functional connectives are treated in a standard manner. A sentence of the form ' $A \geq B$ ' is true at a world x iff the set of worlds where ' A ' is true has greater probability than the set of worlds where ' B ' is true measured by the probability measure associated with x .

The main theorem in /QP/ is that the basic logic outlined above is complete with respect to the set of all probability models, i.e. the theorems of the logic are precisely the set of sentences which are true in all models of this kind. The proof of the completeness theorem proceeds by showing that each non-theorem is falsified in some finite model, using the semantic tools and filtration techniques developed in Hansson and Gärdenfors /28/ and /29/. From this result it follows easily that the logic also is decidable.

This semantic machinery provides a simple way of handling different approaches to the logic of higher order probability statements. The demand that the same probability measure be associated with each world in the model is equivalent to adding an axiom which in essence requires a probability statement to be either necessarily true or necessarily false.

I also outline in the paper how qualitative conditional probability can be treated in much the same way, using Domotor's conditions as a basis for the axioms. The semantics will be similar and a completeness and a decidability theorem can be proved as previously.

Finally, the ordinary necessity operator ' $\Box$ ' is introduced to the language via the definition $\Box A \stackrel{\text{df}}{=} A \sim (B \vee \neg B)$, where ' $\sim$ ' means 'is equally probable as'. The interesting question is then what modal formulas can be derived within our logics. These sets of formulas can be precisely determined using our completeness theorems and well-known features of ordinary modal logics. I propose a construction which starts with a probability model and defines an alternativeness relation ' R ' by ' xRy ' iff $\underline{P}_x(\{y\}) > 0$, where $\underline{P}_x$ is the probability measure associated with the world x .⁴⁴ It is then easy to show that the modal formulas

derivable in the basic probability logic are exactly the theorems in the modal logic $\underline{D}$ (which has been introduced as the appropriate logic for one-place deontic operators). In the logic where probability statements are interpreted analytically, the corresponding modal logic is, not too surprisingly, $\underline{S5}$.

3.

The axioms of a preference relation differ according to the objects among which the relation is supposed to hold. In economic theories, for example, the objects are taken to be commodities or groups of commodities. If one has a philosophic interest in preference logic, it seems proper to let the objects be propositions or sentences. In this section I will discuss preference relations defined on Boolean algebras, the elements of which can be thought of as propositions closed under the ordinary truth-functional operations. I will point out an unsolved problem regarding necessary and sufficient axioms for such a relation to agree with the expected utility of the elements of the Boolean algebra.

For simplicity, I assume that the Boolean algebra is finite, which implies that it can be identified with a subset algebra. I am primarily interested in a binary preference relation $\underline{R}$ among the elements of the algebra. This relation is used to represent the ordering of them according to their expected utility. I will also need another binary relation on these elements, denoted $\underline{\geq}$, which as previously is used to represent the ordering according to the probabilities of the elements.

The first problem I consider is the following: Which conditions on the ordering $\underline{R}$ are necessary and sufficient to guarantee the existence of a utility measure $\underline{u}$ and a probability measure $\underline{P}$ such that the expected utilities mirror preferences. That is to say, for any elements x and y from the algebra, $x \underline{R} y$ iff the expected utility of x is at least as great as the expected utility of y . This problem was posed by Jeffrey⁴⁵ and similar problems have been studied by Bolker /5/.

Although I cannot answer the problem, I do want to point out its importance for the logic of preferences. There are some

trivial necessary axioms for $\underline{R}$ that can be stated, but these are by no means exhaustive. For example, the preference relation has to be transitive and connected and must also satisfy the following averaging condition: If x and y are disjoint elements of the algebra, then their union lies between them in the preference ordering. It may very well be that there is no simple solution to the problem. The axioms may not be formalizable in a first order language.

Jeffrey discusses this problem in /35/ together with the question of the uniqueness of the utility and probability measures obtainable from such an axiomatization. He suggest that the problem can be simplified⁴⁶

"by using two primitives: preference and comparative probability. With these primitives, one ought to be able to drop some of Bolker's restrictions on the algebra of prospects (elements) in favour of conditions on comparative probability and conditions connecting preference and comparative probability. I would expect that in this way one could get significantly closer to an existence theorem in which the conditions are necessary and sufficient for the existence of $\underline{u}$ and $\underline{P}$, ..."

The conditions for this simpler problem involving only comparative probability would be the Kraft-Pratt-Seidenberg axioms as presented earlier. Very little seems to be known about the conditions connecting preferences and probability. As one example of such a condition I can mention the following: If x is disjoint from z and y and if $x \cup z \underline{R} x \cup y$ and $y \underline{R} z$, then $y \geq z$.

The complete answer to this simpler problem is also unknown. Its solution will provide an example of the use of the axioms for qualitative probability and give a strict foundation for a logic of preferences.

NOTES

1. For the connections between choice functions and preference orders, see Hansson /24/ and Houthakker /33/.
2. See Luce and Suppes /43/ for an extensive presentation of the probabilistic approaches.
3. Savage /54/ p. 9.
4. Ibid.
5. See Hansson and Gärdenfors /28/ for references to the major works.
6. This set of states need not be all of $\underline{W}$.
7. The reader is referred to Luce and Suppes /43/ for some of the more important sets of axioms.
8. See Luce and Suppes /43/, p. 285, for the complete list.
9. Jeffrey /34/, p. 73.
10. Luce and Raiffa /41/, p. 13.
11. See Hansson /23/ for a presentation of the main streams.
12. Savage /54/, p. 21.
13. See Fishburn /17/ for a presentation.
14. Any transformation of the form $(a \cdot x + b) / (c \cdot x + d)$, where $a \cdot d - b \cdot c$ is positive, will do.
15. In the first chapter of /2/.
16. Voting functions defined directly as choice functions are studied in Fishburn /18/.
17. See Fine and Fine /14/ and Young /64/.
18. We might as well start with $\underline{R}$ as primitive and then define $\underline{I}$ and $\underline{P}$.
19. See e.g. Fishburn /17/ or Luce and Suppes /43/.
20. See Luce and Suppes /43/ for references.
21. One of the spokesmen for this approach is Cattell /7/.
22. See Luce and Suppes /43/, p. 261.
23. In Lee /39/ several other 'explanations' of this phenomenon are presented.
24. A positive linear transformation is a function of the form $f(x) = a \cdot x + b$, where a is positive.
25. See e.g. Ilmar Waldner "The empirical meaningfulness of interpersonal utility comparisons", Journal of Philosophy, 69 (1972
26. A monotone increasing transformation is a function which satisfies the demand that $x \geq y$ iff $f(x) \geq f(y)$.

27. Arrow /1/, pp. 10 - 11.
28. See Black /3/ for a summary of these papers.
29. Plott /48/ and Fishburn /18/.
30. See e.g. Hansson /26/ and /27/.
31. See Fishburn /18/, Murakami /46/ and Sen /57/ for a presentation of the most important results.
32. Arrow /1/, pp. 94 - 95.
33. This is the so called weak independence condition studied in Hansson /27/.
34. See e.g. Ramstedt /51/ for such a misinterpretation.
35. In Hansson /25/.
36. This program is carried out in greater detail in Gärdenfors /20/.
37. See McVitie and Wilson /44/ and /45/.
38. Not to be confused with the stability condition in /PVF/.
39. See Luce and Raiffa /41/, pp. 366 - 367, or Steinhaus /58/.
40. Formally, the utility measure is a function from the outcomes to real numbers.
41. The paper by de Finetti which is easiest to find is in Kyburg and Smokler /38/.
42. Savage /54/, p. 41.
43. If the probability measure is the same for all worlds, it turns out that higher order statements become analytical.
44. The relation R is used here as in standard semantics for modal logics (see Hansson and Gärdenfors /28/).
45. In Jeffrey /34/, one of the assumptions for his uniqueness theorem (p. 119) is that the utility and probability measures exist. There he does not discuss the possibility of replacing this condition with a set of axioms on the relations $\underline{R}$ and $\underline{\geq}$. This problem is explicitly stated in Jeffrey /35/.
46. Jeffrey /35/, pp. 77 - 78.

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STUDENTLITTERATUR
LUND SWEDEN